Journal of Pure and Applied Algebra 30 (1983) 23-38
North-Holland 

# THE FUNDAMENTAL GROUP OF A TOPOS

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Communicated by M. Barr
Received 17 September 1981
Revised 14 February 1983

## 1. Introduction

The profinite fundamental group of a topos was defined by Grothendieck and used for algebraic geometry, [2], [4, pp. 285–290]. It resembles the fundamental group of a topological space. For example, if the space $X$ is *well-connected* (i.e., nice enough to have a universal, simply connected covering), then Grothendieck's fundamental group for the topos of sheaves over $X$ is the *profinite completion* of the usual fundamental group of $X$. In this paper we define a new fundamental group for a topos which is internal (as a pro-group) and which corresponds to the actual fundamental group rather than the profinite completion. Its main features are:

(1) Our fundamental group is defined internally. It does not depend on choices (such as the choice of a base point) and is even defined for topoi without points. Similarly it is functorial with respect to all geometric morphisms between topoi, not just 'base-point preserving' ones.

(2) It is not required that the topos be connected. The fundamental group behaves differently on different components.

(3) The traditional fundamental group of a well-connected topological space can be recovered from our definition applied to the topos of sheaves. Similarly if $G$ is a group, then $G$ (regarded as a pro-group) is the fundamental group of the topos of $G$-sets (i.e., sets on which $G$ acts) [Grothendieck's definition yields the profinite completion from which $G$ and the traditional fundamental group cannot be recovered.]

(4) If we are considering a connected topos with a point, then our fundamental group can be pulled back along the point to a pro-group in Sets. This pro-group has a profinite completion which is isomorphic to the Grothendieck fundamental group. There is an internal profinite completion which can be viewed as an internal point-free version of the Grothendieck group. The use of pro-groups is analogous to the internal Galois group which is an internal pro-group (or pro-groupoid) rather than

* The author thanks the NSF for support under grant MCS-8002166.

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an internal group (see [7], [8]). The Galois groups are always profinite (i.e., equal to their profinite completions).

(5) Our best results are for the profinite completion of the fundamental group. For example, there is a profinite version of the Hurewicz theorem (see *Theorem 3.9*). Analogous questions for the pro-group remain open. A disappointment is the inability *so far* to locate the universal cover where it 'should be' (in the topos of $\pi$-actions where $\pi$ is the fundamental group and covers, see [1]. The relationship between that paper and this one also remains open.

## 1. The fundamental group as a pro-group

The fundamental group of a topos seems to be a pro-group rather than a group.

**Definition.** Let Grp be the category of ordinary (i.e., set-based) groups. Then a *pro-group* $P$ in a topos $\mathcal{E}$ is a left exact (=finite limit preserving) functor:

$$P: \mathrm{Grp} \to \mathcal{E}.$$

For example every internal group $H$ in $\mathcal{E}$ corresponds to a pro-group $\mathrm{Hom}(H, -)$ but most pro-groups are not of this form.

**Definition.** Let $G$ be a group in Sets and let $\mathcal{E}$ be a topos. By a *generalized G-Torsor* $T$ in $\mathcal{E}$ we mean an object $T$ on which $G$ acts (on the left) so that the map $G\times T \to T\times T$ [given by $(g,t)\to(gt,t)$] is an isomorphism and so that $T$ has *clopen extent* (meaning that the image of $T$ under $T\to 1$ is a *complemented* subobject of 1). $T$ is a $G$-Torsor in the ordinary sense iff the extent of $T$ is 1. If $\mathcal{E}$ is connected, then (aside from the trivial object, 0) all generalized $G$-Torsors are $G$-Torsors in the usual sense.

**Definition.** Let $\mathcal{E}$ be a topos. For each group $G$ in Sets let $\pi(G)$ be the *colimit* of the diagram of generalized $G$-torsors and $G$-equivariant maps. (If this colimit does not exist, then $\mathcal{E}$ does not have a fundamental group. If $\mathcal{E}$ is a Grothendieck topos the diagram is essentially small and the colimit does exist.) Note that $\pi(G)$ is defined as a colimit of an *externally* defined diagram. The internal colimit would be trivial since, locally, all Torsors look alike. Some external aspect is needed in defining $\pi$ since it is supposed to measure how well local phenomena can be patched together to form global phenomena.

We further regard $\pi$ as a functor as follows: If $u: G\to H$ is a group homomorphism and if $T$ is a generalized $G$-Torsor, then $u^*(G)=H\otimes T$ (tensoring over $G$) is a generalized $H$-Torsor. This enables us to map $\pi(G)$ to $\pi(H)$.

**Notation.** If $\mathcal{E}$ is a topos, then $\pi_{\mathcal{E}}$ or even $\pi(\mathcal{E})$ will often be used to denote the *fundamental group* of $\mathcal{E}$. The simpler notation $\pi$ will be used when the topos involved is understood. The distinction between $\pi(\mathscr{E})$ (the fundamental group of $\mathscr{E}$) and $\pi(G)$ (the value of the functor $\pi$ at the group $G$) should always be clear from the context.

**Remark.** The best way to understand this definition and see that it does the 'right thing' is to consider some examples. These are presented at the end of this section after first verifying that $\pi$ is left exact and well-behaved with respect to geometric morphisms.

**Proposition 1.1.** *The functor $\pi$ is left exact (and so is a pro-group).*

**Proof.** That $\pi$ is functorial and finite product preserving and preserves the terminal object, 1, is all straightforward. It remains to show that $\pi$ preserves equalizers. Let $u$ and $v$ from $G$ to $H$ be homomorphisms and let $E \subseteq G$ be their equalizer. Let $T$ be a $G$-torsor and let $H \otimes_u T$ and $H \otimes_v T$ denote $u^{\#}(T)$ and $v^{\#}(T)$ respectively. Let

$$\lambda : H \otimes_u T \to H \otimes_v T$$

be a global $H$-equivariant map. Define:

$$T_1 = \{t \in T \mid \lambda(1 \otimes_u t) = 1 \otimes_v t\}$$

(Note that $T_1$ is actually defined in $\mathscr{E}$ as an equalizer and the 'set-theoretic' notation is only suggestive.) It is clear that $T_1$ is closed under the action of $E$. Moreover if $g \in G$ but $g \notin E$, then $g(T_1)$ is disjoint from $T_1$. These facts show that $T_1$ is (locally) either empty or an $E$-torsor. To show that $T_1$ is a generalized $E$-torsor, it remains to prove that $T_1$ has clopen extent. For each $h \in H$ define:

$$T_h = \{t \in T \mid \lambda(1 \otimes_u t) = h \otimes_v t\}.$$

Clearly $T$ is the disjoint union of $\{T_h\}$ as $h$ varies in $H$ (recall that $H$ is a group in Sets). Let $S \subseteq T$ be the union of $\{g(T_1)\}$ for $g \in G$. Since $g(T_1)$ is just $T_h$ for $h = u(g)v^{-1}(g)$ it follows that $S$ is complemented. Moreover if $p : T \to 1$ is the unique map to 1, then $p(S) = p(T_1)$ and $S = p^{-1}p(S)$. Since $T$ has clopen extent, it readily follows that $p(S)$ is the clopen extent of $T_1$. So $T_1$ is a generalized $E$-torsor and its existence shows that $\pi(\mathscr{E})$ maps onto the equalizer of $\pi(u)$ and $\pi(v)$. That $\pi$ preserves monos is straightforward, which completes the proof.

**Proposition 1.2.** *Let $\mathscr{E}$ and $\mathscr{F}$ be topoi and let $r^* : \mathscr{E} \to \mathscr{F}$ be an inverse image functor. Then the composition $r^*\pi(\mathscr{E})$ is a pro-group in $\mathscr{F}$. There is an associated natural transformation $\pi(r^*)$ from $r^*\pi(\mathscr{E})$ to $\pi(\mathscr{F})$ which is functorial in the sense that if $s^* : \mathscr{F} \to \mathscr{G}$ is also an inverse image functor, then*

$$\pi(s^*)s^*(\pi(r^*)) = \pi(s^*r^*).$$

**Proof.** For each group $G$ we have defined $\pi(\mathscr{E})(G)$ as a colimit of $G$-torsors. Since

$r^*$ preserves $G$-torsors and preserves colimits, there is an obvious map from $r^*\pi(\mathcal{E})(G)$ into $\pi(\mathcal{F})(G)$. It is straightforward to verify the details that this is natural and interacts properly with $s^*$.

**Remark.** From the above result, $\pi(\mathcal{E})$ varies covariantly with inverse image functors, hence contravariantly with geometric morphisms and therefore contravariantly with continuous maps in the case of spatial topoi. This may seem unexpected since the topological fundamental group is covariant. However, if we recall that $\pi(\mathcal{E})$ is a pro-group generalizing the functor $\mathrm{Hom}(\pi_1(X), -)$ we see that group homomorphisms correspond to natural transformations in the opposite direction.

**Example 1** ($H$-sets). Let $H$ be a group and consider the topos of $H$-sets. It is convenient to think of an $H$-set as a set together with a *right* action by $H$ while $G$-torsors in $H$-sets will have a compatible *left* action by $G$. We claim that the colimit of all $G$-torsors in $\mathrm{Hom}(H, G)$ and so $\pi(H\text{-sets})$ is essentially $H$ (or more precisely, the pro-group $\mathrm{Hom}(H, -)$.) An element of the colimit is represented by a $G$-torsor $T_0$ together with a point $t_0 \in T_0$. This produces a map $m : H \to G$ defined so that $t_0 h = m(h) t_0$. Conversely, given $m : H \to G$ then $G$ acquires a right $H$-action (via $m$) and becomes a $G$-torsor (under left multiplication) with a distinguished element (the identity) which corresponds to $m$. From this $\pi(G)$ is readily shown to be $\mathrm{Hom}(H, G)$. The $H$-action on $\pi(G)$ can be shown to be by conjugation in the sense that $(mb)(x) = m(hxh^{-1})$.

**Remark.** If $G$ is abelian, then $\pi(G)$ is simply the set of all equivalence classes of $G$-torsors. This happens because in the colimit each $G$-torsor gets identified to a point. However, if $G$ is *not* abelian, then multiplication by $g$ need not be $G$-equivariant for every $G$-torsor. So the $G$-torsor $T$ need not collapse into a single point. In the above example, the $G$-torsor $T$ is mapped in the colimit $\pi(G)$ onto a $G$-conjugacy class of maps from $H$ to $G$. This topic is pursued further in the discussion of the relation between the fundamental group and homology (see Theorem 3.9).

**Example 2** (*Universal Covers*). Let $X$ be a connected topological space which is nice enough to have a universal covering $p : X^* \to X$. Let $\pi_1$ be the deck translation group which then acts transitively on each fibre $p^{-1}(x)$ (by universality). Pick a base point $x_0^*$ in $X^*$ and let $x_0 = p(x_0^*)$. Let $T$ be any $G$-torsor sheaf over $X$ and let $t_0$ in $T$ be a given point lying over $x_0$ in $X$. By universality, there exists a map $\alpha : X^* \to T$ which sends $x_0^*$ to $t_0$. For each deck translation $d$ in $\pi_1$ there is a unique $g \in G$ such that $\alpha d(x_0^*) = g t_0$. It follows that $\alpha d = g\alpha$ (as they agree at $x_0^*$). This sets up a group homomorphism $m : \pi_1 \to G$ where $m(d)$ is that $g$ for which $\alpha d = g\alpha$. Each $m : \pi_1 \to C$ arises in this manner since from $m$ we get a $G$-torsor $G \otimes X^*$. It follows that each stalk of the sheaf $\pi(G)$ is in one-one correspondence with $\mathrm{Hom}(\pi_1, G)$ so the pro-group $\pi$ is a functor which is *locally* represented by the deck translation group $\pi_1$. The global representation of $\pi$ is described in the final section of this paper (Example 4.1).

**Example 3** (*Pro-groups in Sets*). (a) Every group $H$ defines a pro-group by the representable functor Hom($H$, –). Those pro-groups $P$: Grp→Sets which are proper and limit-preserving (i.e., preserve all limits, not just the finite ones) are (to within natural equivalence) precisely those of the form Hom($H$, –), by Freyd's theorem. It can also be shown that $H$ is determined within isomorphism by $P$.

(b) Every topological group $H$ defines a pro-group Hom($H$, –) where Hom($H$, $G$) is the set of continuous homomorphism (taking $G$ to be discrete). Those proper functors $P$: Grp→Sets which are not only left exact but also preserve infinite intersections (of subgroups) are precisely those respresentable as hom($H$, –) for a *localic* group $H$. [This result corrects a mistaken claim that topological groups sufficed. The improvement was suggested by John Isbell, the proof (not included here) was obtained in joint work with David Joyce.]

(c) If $H$ is a topological group and $\{H_i\}$ is a filter of sub-groups, then the functor $P$: Grp→Sets defined by:

$$P(G) = \operatorname{Colim} \operatorname{Hom}(H_i, G)$$

is left exact. Conversely every proper pro-group is represented by such a filter (which is constructible from the canonical diagram). Non-proper pro-groups exist (Isbell) and they can be visualized as above in terms of 'big' topological groups $H$ which exist in a larger universe.

**Remark.** If $\Gamma$ is a proper pro-group in Sets, then we can construct the topos Sets$^\Gamma$ of sets on which $\Gamma$ acts (see next section). The question of whether $\pi(\text{Sets}^\Gamma)$ is $\Gamma$ remains open (an apparent counter-example found previously does not work). For *profinite groups* $\Gamma$ we do have $\pi(\text{Sets}^\Gamma) = \Gamma$ and a generalization to $\mathcal{E}^\Gamma$ (see Section 3).

## 2. On pro-groups in a topos

This section contains some technical points which will be used later.

### *Γ-actions*

Let $\Gamma$: Grp→$\mathcal{E}$ be a pro-group which is proper (as defined below). Then if $\mathcal{E}$ has enough limits, we can define what it means for $\Gamma$ to act and we can construct the topos $\mathcal{E}^\Gamma$ of all $\Gamma$-actions and $\Gamma$-equivariant maps.

**Definition.** By an *action*, $(A, G)$, we mean a set $A$ and a group $G$ (in Sets) which acts on $A$. The pair $(m, s)$ is an *action map* from $(A, G)$ to $(B, H)$ if $m: A \to B$ and $s: G \to H$ is a function such that $m(sha) = h(ma)$ for all $h \in H$ and all $a \in A$. We denote

Act = the category of actions.

Observe that if Grp is the category of groups in Sets, then there is an obvious injection:

$$I: \mathrm{Grp} \to \mathrm{Act}^{\mathrm{op}} \quad \text{where } I(G) = (\emptyset, G).$$

If $\Gamma: \mathrm{Grp} \to \mathscr{E}$ is a pro-group and if $\Gamma$ 'acts on $W$', then conceptually there should be a functor $W$ from $\mathrm{Act}^{\mathrm{op}}$ to $\mathscr{E}$ which sends $(A, G)$ to the object of action maps from $(W, \Gamma)$ to $(A, G)$. To make this precise, we need to take care of a cardinality condition. Recall that the infinite cardinal $m$ is *regular* if $n_j < m$ for all $j \in J$ and $\mathrm{Card}(J) < m$ imply $\sum n_j < m$. (All infinite, non-limit cardinals are regular.)

**Definition.** Let $m$ be a regular infinite cardinal and let $\mathscr{E}$ be a topos. Let $m$-Grp be the category of all groups with cardinal less than $m$. We say that $\Gamma: \mathrm{Grp} \to \mathscr{E}$ is *m-proper* if $\Gamma$ is the left Kan extension of its restriction to $m$-Grp.

Since left Kan extensions from $m$-Grp preserve left exactness, the category of *m-proper pro-groups* is essentially the category of left exact functors from $m$-Grp. (For $\mathscr{E}$ = Sets, a pro-group $\Gamma$ is proper iff it is $m$-proper for some regular $m$.)

**Definition.** Let $\mathscr{E}$ be a topos and let $\Gamma$ from $m$-Grp to $\mathscr{E}$ be an $m$-proper pro-group for $m$ some regular cardinal. Assume that $\mathscr{E}$ has limits of all diagrams of cardinal $2^n$ whenever $n < m$. Let $m$-Act be the category of all actions $(A, G)$ with Card $A$ and Card $G$ less than $m$. Let $(m\text{-Sets})^G$ be the category of all sets $A$ (of cardinal less than $m$) such that $G$ acts on $A$. The maps of $(m\text{-Sets})^G$ are to be $G$-equivariant. So if $\mathrm{Card}(G) < m$, then

$$(m\text{-Sets})^G \subseteq m\text{-Act}.$$

Note also that $I: m\text{-Grp} \to m\text{-Act}^{\mathrm{op}}$.

Define $\mathscr{E}^\Gamma$ as the category of all functors:

$$W: (m\text{-Act})^{\mathrm{op}} \to \mathscr{E}$$

such that:

(1) $WI = \Gamma$ (so $W(\emptyset, G) = \Gamma(G)$ and similarly for maps).
(2) $W$ is left exact.
(3) The restriction of $W$ to $(m\text{-Sets}^G)^{\mathrm{op}}$ preserves *all* limits, as a functor to $\mathscr{E}_{\Gamma(G)}$.
(4) The morphisms of $\mathscr{E}^\Gamma$ are natural transformations over $I$.

**Theorem 2.1.** *With the above assumptions, $\mathscr{E}^\Gamma$ is a topos.*

**Proof.** We shall only show how to modify the lengthy argument given in the proof of Theorem 1.1 of [8] so that it applies here. First $W$ is determined by its values $W(G, G)$ for each $G$ in $m$-Grp, where $G$ acts on itself by left multiplication. This follows from the canonical colimit construction given in [8], which gives us a limit in $(m\text{-Sets}^G)^{\mathrm{op}}$. We can now apply the argument of [8], which placed the objects

$\{W(G,G)\}$ into a Wraith glueing construction, except for the technical point of Lemma 4.1 of [8] which relied heavily on finiteness. However, this lemma could have been proved more easily using an adjoint transpose (of $\zeta$ with respect to $h^0, h_0$). Then finiteness is not needed.

**Remark.** Theorem 2.1 extends to pro-categories. The proof of theorem 1.1 of [8], as modified above, applies.

**Notation.** Let $\mathscr{E}$ be a topos. Then Pro-Grp($\mathscr{E}$) is the *dual* of the category of left exact functors from Grp to $\mathscr{E}$.

*Products and coproducts*

Let $\Gamma$: Grp$\to\mathscr{E}$ and $\Delta$ : Grp$\to\mathscr{E}$ be in Pro-Grp($\mathscr{E}$). We define their *coproduct* $\Gamma+\Delta$ so that $(\Gamma+\delta)(G)$ is $\Gamma(G)\times\Delta(G)$. This is easily seen to be their coproduct in Pro-Grp($\mathscr{E}$).

The product of $\Gamma$ and $\Delta$ in Pro-Grp($\mathscr{E}$) is more difficult. If $\mathscr{E}$ = Sets and if $\Gamma$ and $\Delta$ are representable, this corresponds to defining Hom$(\Gamma\times\Delta, G)$ in terms of Hom$(\Gamma, G)$ and Hom$(\Delta, G)$. It is the subset of Hom$(\Gamma, G)\times$Hom$(\Delta, G)$ consisting of pairs of maps whose 'ranges commute'.

**Definition.** Let $G$ be a group in Sets. We call $(G_1, G_2)$ a *commuting pair* of subgroups of $G$ if $G_1\subseteq G$, $G_2\subseteq G$ and every $x\in G_1$ commutes with $y\in G_2$.

Let $\Gamma$ and $\Delta$ be in Pro-Grp($\mathscr{E}$). We define their product, $\Gamma\times\Delta$, by

$$(\Gamma\times\Delta)(G)=\bigcup\{\Gamma(G_1)\times\Delta(G_2)\mid (G_1,G_2) \text{ is a commuting pair}\}.$$

(We regard each $\Gamma(G_1)\times\Delta(G_2)$ as a subobject of $\Gamma(G)\times\Delta(G)$ and the union is a union of subobjects. The obvious map $(\Gamma\times\Delta)(G)\to\Gamma(G)\times\Delta(G)$ is a natural transformation, corresponding to a map $\Gamma+\Delta\to\Gamma\times\Delta$ in Pro-Grp($\mathscr{E}$).)

**Proposition 2.2.** *$\Gamma\times\Delta$ as defined above is a product in* Pro-Grp($\mathscr{E}$).

**Proof.** First, a direct proof shows that $\Gamma\times\Delta$ is a left exact functor. Next, we must first define the projection maps $p_1:\Gamma\times\Delta\to\Gamma$ and $p_2:\Gamma\times\Delta\to\Delta$. These are to be natural transformations, with $p_1(G):\Gamma(G)\to(\Gamma\times\Delta)(G)$ and $p_2(G):\Delta(G)\to(\Gamma\times\Gamma)(G)$. Since $(G,1)$ is a commuting pair and since $\Delta(1)=1$, as $\Delta$ is left exact, the definition of $p_1(G)$ is obvious. Similarly $p_2(G)$ is defined using the commuting pair $(1,G)$.

Let $\Psi\in$ Pro-Grp($\mathscr{E}$) and maps $a:\Psi\to\Gamma$, $b:\Psi\to\Delta$ be given. We must find an appropriate map $c:\Psi\to\Gamma\times\Delta$. Note that $a(G):\Gamma(G)\to\Psi(G)$ and $b(G):\Delta(G)\to\Psi(G)$ and we must define $c(G):(\Gamma\times\Delta)(G)\to\Psi(G)$. Clearly $a(G)\times b(G)$ defines a map from $(\Gamma\times\Delta)(G)$ to $(\Psi\times\Psi)(G)$. It suffices to compose this with the *diagonal map* $(\Psi\times\Psi)(G)\to\Psi(G)$. But if $(G_1,G_2)$ is a commuting pair, then there is a group homomorphism $m:G_1\times G_2\to G$ given by group multiplication. Therefore $\Psi(m)$ maps $\Psi(G_1)\times\Psi(G_2)$ to $\Psi(G)$ (since $\Psi(G_1\times G_2)=\Psi(G_1)\times\Psi(G_2)$) and the maps $\Psi(m)$ clearly patch together correctly to give us the desired diagonal map.

Similarly we can show that

$$c(G):(\Gamma\times\Delta)(G)\to\Psi(G)$$

is uniquely determined by the naturality of $c$ and the requirements that $cp_1=a$ and $cp_2=b$. It suffices to consider a commuting pair $G_1,G_2$ of subgroups of $G$ and examine the restriction of $c(G)$ to $\Gamma(G_1)\times\Delta(G_2)$. Let $H=G_1\times G_2$ and $H_1=G_1\times\{1\}$ and $H_2=\{1\}\times G_2$. Let $m:H\to G$ denote group multiplication, a homomorphism as $G_1,G_2$ commute. Let $\pi_i:H\to G$ be the projection maps for $i=1,2$. To show that $c(G)$ restricted to $\Gamma(G_1)\times\Delta(G_2)$ is uniquely determined it suffices to show that $c(H)$ restricted to $\Gamma(H_1)\times\Delta(H_2)$ is uniquely determined and apply the naturality of $c$ to the map $m$. For this it suffices to show that $\Psi(\pi_i)c(H)$ restricted to $\Gamma(H_1)\times\Delta(H_2)$ is determined for $i=1,2$ but this is straightforward.

## 3. The profinite and restricted fundamental groups

A topos $\mathscr{E}$ may fail to have a fundamental group because it lacks enough colimits, such as the colimit of all $G$-torsors where $G$ is a group of large cardinal. For such topoi it might be best to restrict our definition to groups of cardinal less than $m$ where $m$ is a regular cardinal. In particular we consider the restriction to finite groups (other restrictions could presumably be treated similarly). This produces the notion of the *profinite fundamental group* which (Theorem 3.7) internalizes Grothendieck's fundamental group and which has good properties (e.g., Theorem 3.3, Proposition 3.8, Theorem 3.9 below).

**Definition.** Recall that a profinite group in $\mathscr{E}$ is defined to be a left exact functor from Fin Grp (finite groups) to $\mathscr{E}$.

Define $\hat{\pi}_{\mathscr{E}}$ : Fin Grp$\to\mathscr{E}$ as the restriction of $\pi_{\mathscr{E}}$. Then $\hat{\pi}_{\mathscr{E}}$ (or just $\hat{\pi}$) is the *Profinite Fundamental Group* of $\mathscr{E}$. Note that Propositions 1.1 and 1.2 also apply to $\hat{\pi}$.

**Examples.** $\hat{\pi}$ is the profinite completion of $\pi$. So for $H$-sets, $\hat{\pi}$ is (the functor represented by) the profinite completion of $H$. Similarly, for spatial topoi, over well-connected topological spaces, $\hat{\pi}$ is the profinite completion of the traditional fundamental group.

**Notation.** We let Profin Grp($\mathscr{E}$) be the category of profinite groups in $\mathscr{E}$ (and the *dual* of natural transformations). If $\mathscr{E}$ has enough colimits, then Profin Grp($\mathscr{E}$) can be regarded as the full subcategory of Pro-Grp($\mathscr{E}$) comprised of those pro-groups which are $\aleph_0$-proper (see previous section). The inclusion of Profin Grp($\mathscr{E}$) in Pro-Grp($\mathscr{E}$) then has a left adjoint which assigns to each pro-group its profinite completion (which is simply its restriction to Fin Grp).

For technical reasons we need the following definition.

**Definition.** Let $\Gamma$ be in Profin Grp($\mathscr{E}$), let $G$ be a finite group in Sets, and let $g \in G$ be given. The *nullity* of $g$ is, conceptually, the set of all $\gamma \in \Gamma(G)$ for which $g$ is not in 'the range of $\gamma$'. We define:

$$N_\Gamma(g) = \bigcup\{\Gamma(K) \mid K \subseteq G \text{ and } g \notin K\}.$$

Note that we regard $\Gamma(K) \subseteq \Gamma(G)$ when $K \subseteq G$ and $N_\Gamma(g)$ is a union of subobjects of $\Gamma(G)$. We let $N(g) = N_\Gamma(g)$ when $\Gamma$ is understood. Note that $N(g)$ is geometrically defined despite the negation '$g \notin K$' since $N(g)$ is a union of specified subobjects.

**Notation.** For each $g \in G$ let $\bar{g} : G \to G$ represent the group homomorphism of conjugation by $g$.

**Proposition 3.1.** *Let $\Gamma$ and $\Delta$ be in* Profin Grp($\mathscr{E}$). *Recall that $(\Gamma \times \Delta)(G)$ is defined as a subobject of $\Gamma(G) \times \Delta(G)$. It can alternatively be defined by*

$$(\Gamma \times \Delta)(G) = \{(\gamma, \delta) \in \Gamma(G) \times \Delta(G) \mid \Gamma(\bar{g})(\gamma) = \gamma \text{ or } \delta \in N_\Delta(g) \text{ for all } g \in G\}.$$

**Proof.** Since $G$ is finite this is a proposed geometric description of $(\Gamma \times \Delta)(G)$. If $\mathscr{E}$ = Sets, then $\Gamma$ and $\Delta$ can be regarded as ordinary profinite groups and $\gamma : \Gamma \to G$, $\delta : \Delta \to G$ as maps. The condition says that either $g\gamma(x)g^{-1} = \gamma(x)$ for all $x \in \Gamma$ or $g$ is not in the range of $\delta$. So if $g$ is in the range of $\delta$, then $g\gamma(x) = \gamma(x)g$ for all $x \in \Gamma$. Using the results of [11], this proves the result for a topos $\mathscr{E}$ (with countable limits).

**Remark.** Let $\Gamma$ be an ordinary profinite group and let $G$ be a finite group in Sets. Let $\Gamma(G)$ be the set of continuous homomorphisms from $\Gamma$ to $G$. Then $\Gamma$ acts on $\Gamma(G)$ by conjugation. [If $x \in \Gamma$ and $\gamma : \Gamma \to G$ let $(x\gamma)(y) = \gamma(x^{-1}yx)$.] Since $\Gamma(G) \in \text{Sets}^\Gamma$ we can lift $\Gamma$ to $\bar{\Gamma}$ in Profin Grp($\text{Sets}^\Gamma$).

**Proposition 3.2.** *Let $\Gamma$ in* Profin Grp($\mathscr{E}$) *be given. Then $\Gamma$ can be lifted to $\bar{\Gamma}$ in* Profin Grp($\mathscr{E}^\Gamma$) *by a geometric construction which extends the above definition of $\bar{\Gamma}$ for $\mathscr{E}$* = Sets. *If $U^* : \mathscr{E}^\Gamma \to \mathscr{E}$ is the 'underlying object functor' of the inverse image functor defined in* [8, *Lemma* 3.4], *then $U^*\bar{\Gamma} = \Gamma$.*

*$\Gamma$ has a different lifting to* Profin Grp($\mathscr{E}^\Gamma$) *obtained by composing with the 'constant action' functor $\mathscr{E} \to \mathscr{E}^\Gamma$. When there is no danger of confusion, we let $\Gamma$ also denote this profinite group in $\mathscr{E}^\Gamma$ with trivial action.*

**Proof.** Let $\Gamma$ be given and let $G$ be a finite group. To show that $\Gamma(G)$ lies in $\mathscr{E}^\Gamma$ we have to regard $\bar{\Gamma}(G)$ as a left exact functor from the category of finite group actions to $\mathscr{E}$ (see [7]). If the finite group $H$ acts on the finite set $A$, then $\bar{\Gamma}(G)(A, H)$ is, conceptually, the object of action maps from $(A, H)$ to $(\Gamma(G), \Gamma)$ where $\Gamma$ acts on $\Gamma(G)$ by conjugation. These maps are envisioned as pairs $(m,s)$ where $m:A\to\Gamma(G)$ and $s\in\Gamma(H)$ such that for $x\in\Gamma$ and $a\in A$ we have

$$m(s(x)a)=x(ma)=g(ma)g^{-1} \quad \text{where } g=m(a)(x)^{-1}.$$

Recall that $\bar{g}:G\to G$ is defined by $\bar{g}(j)=gfg^{-1}$. In order for $(m,s)$ to be an action map we must require that $m(ha)=\Gamma(\bar{g})m(a)$ whenever $a\in A$ and $(g^{-1},h)$ is in the range of $(ma,s):\Gamma\to G\times H$. Therefore, we define:

$$\bar{\Gamma}(G)(A,H)=\{(m,s)\mid m:A\to\Gamma(G),\ s\in\Gamma(H) \text{ and for all } a\in A,\ g\in G,$$
$$h\in H,\ m(ha)=\Gamma\bar{g})ma \text{ or } (ma,s)\in N(g^{-1},h)\}.$$

Note that $(ma,s)\in\Gamma(G\times H)$, essentially because $\Gamma$ is left exact and that $(g^{-1},h)\in G\times H$.

Since $A$, $G$ and $H$ are all finite, this definition is geometric, and, from the above discussion $\bar{\Gamma}(G)(A,H)$ has the appropriate properties in the case $\mathscr{E}=$ Sets.

**Theorem 3.3.** *Let $\Gamma$ be in* Profin Grp($\mathscr{E}$). *Then*

$$\hat{\pi}(\mathscr{E}^{\Gamma})=\hat{\pi}(\mathscr{E})\times\bar{\Gamma}.$$

(*Note that $\bar{\Gamma}$ is defined above and that $\hat{\pi}(\mathscr{E})$ in* Profin Grp($\mathscr{E}$) *is lifted to* Profin Grp($\mathscr{E}^{\Gamma}$) *by using the trivial $\Gamma$-action. The product of these profinite groups in $\mathscr{E}^{\Gamma}$ is defined above.*)

**Proof.** We need to relate generalized torsors in $\mathscr{E}$ to generalized torsors in $\mathscr{E}^{\Gamma}$. For convenience we shall work with actual torsors, the extension to generalized torsors being straightforward.

Suppose that $T$ is a $G$-torsor in $\mathscr{E}$. If we want to define a '$G$-preserving $\Gamma$-action' on $T$, then, conceptually, there should be a map $\sigma:T\to\Gamma(G)$ such that for $t\in T$ we have $\sigma(t):\Gamma\to G$ sends $x\in\Gamma$ to the unique $g$ in $G$ for which $xt=g^{-1}t$. As a consequence $\sigma(g_0t)$ would presumably be $\Gamma(\bar{g}_0)\sigma(t)$ (where $\Gamma(\bar{g}_0)$ is composition with conjugation by $g_0$). To make this precise we need the following lemmas.

**Lemma 3.4.** *Let $T$ be a $G$-Torsor in $\mathscr{E}$ (where $G$ is a finite group in* Sets). *Let $\Gamma$ be in* Profin Grp($\mathscr{E}$) *and let $\sigma:T\to\Gamma(G)$ be defined so that $\sigma(gt)=\Gamma(\bar{g})\sigma(t)$. Then $(T,\sigma)$ can be regarded, in a natural way, as a $G$-Torsor in $\mathscr{E}^{\Gamma}$.*

**Proof.** To place $(T,\sigma)$ in $\mathscr{E}^{\Gamma}$ we must interpret $(T,\sigma)$ as a left exact functor from the category of finite group actions to $\mathscr{E}$ (see [8]). Suppose that the finite group $H$ acts on the finite set $A$. Then $(T,\sigma)(A,H)$ is, in concept, the object of action maps from $(A,H)$ to $(T,\Gamma)$ where $\Gamma$ 'acts' on $T$ via $\sigma$ so that if '$x\in\Gamma$', then $xt=g^{-1}t$ where $g=\sigma(t)(x)$. Therefore, an action map should (presumably) consist of a function $m:A\to T$ together with $s:\Gamma\to H$ such that for all $x\in\Gamma$ we have $m(s\times a)=xma$. Imagine that $\sigma(ma)x=g^{-1}$; then $xma=gma$. If, in addition, $s(x)=h$, then

$m(s\times a)=m(ha)$. So if $(g^{-1},h)$ in $G\times H$ is in the range of $(\sigma ma, s)$ from $\Gamma$ to $G\times H$, then $m(ha)$ should be $gma$. To make this precise:

$$(T,\sigma)(A,H)=\{m:A\to T,\ s\in\Gamma(H)\,|\,\text{for all } g\in H,\ h\in H,\ a\in A:$$
$$\text{either } m(ha)=g(ma) \text{ or } (\sigma ma, s)\in N(g^{-1},h)\}.$$

It remains to show that $(T,\sigma)$ is functorial, left exact and a $G$-Torsor in $\mathscr{E}^{\Gamma}$. This is all straightforward, most of the verifications being reducible to the case $\mathscr{E}=$ Sets, using the results of [11].

**Lemma 3.5.** *Conversely, every G-torsor $\bar{T}$ in $\mathscr{E}^{\Gamma}$ is equivalent to a Torsor of the form $(T,\sigma)$ as described above where $T=U^*(\bar{T})$. [Recall, from* [8], *that $U^*:\mathscr{E}^{\Gamma}\to\mathscr{E}$ is conceived of as the 'underlying object functor'.*]

**Proof.** $U^*(\bar{T})$ is defined as an equivalence class of pairs $(a,x)$ where $x\in\bar{T}(A,H)$ and $a\in A$, see [6]. Let $[a,x]$ denote the equivalence class containing $(a,x)$. $G$ acts on $\bar{T}$ so for each $g\in G$ there is a natural map, which shall also be denoted by $g$, from $\bar{T}(A,H)$ to $\bar{T}(A,H)$. Then $G$ also acts on $U^*(\bar{T})$ by sending $[a,x]$ to $[a,gx]$. Since $\bar{T}$ is a $G$-Torsor, given any $(a,x)$, the collection $[a,gx]$ exhausts $U^*(\bar{T})$. So for each $h\in H$ there is a unique $g\in G$ such that $[ha,x]=[a,g^{-1}x]$, this defines a *function* (not necessarily a homomorphism) $f(a,x):H\to G$. We define $(a,x)$ to be *regular* if $f(a,x)$ is a group homomorphism. When $(a,x)$ is regular we can determine a member of $\Gamma(G)$ as follows: There is a projection $p:\bar{T}(A,H)\to\Gamma(H)$, let $s=p(x)$. Then, if $f=f(a,x)$, we have $\Gamma(f)(s)\in\Gamma(G)$. It remains to show that every member of $U^*(\bar{T})$ can be represented by a regular $(a,x)$ and that, for regular $(a,x)$, the element $\Gamma(f)(s)$ depends only on the equivalence class $[a,x]$ and that this defines the required map $\sigma:U^*(\bar{T})\to\Gamma(G)$.

It suffices to do all of this, geometrically, in the case $\mathscr{E}=$ Sets. Let $\Gamma$ be a profinite group in Sets and let $\bar{T}$ be a $G$-Torsor on which $\Gamma$ acts, so that the $G$-action preserves the $\Gamma$-action. Let $T$ be the underlying set. Then $\bar{T}(A,H)$ is the set of action maps from $(A,H)$ to $(T,\Gamma)$. If $x\in\bar{T}(A,H)$ then $x$ consists of a map $m:A\to T$ and $s\in\Gamma(H)$ for which $m(s(\gamma)a)=\gamma m(a)$ for all $a\in A$ and all $\gamma\in\Gamma$. The pair $[a,x]$ corresponds to $m(a)\in T$. (So $U^*(\bar{T})=T$ in this way.) Define $\sigma:T\to\Gamma(G)$ so that $\sigma(t)(\gamma)=g$ iff $\gamma t=g^{-1}t$. Let $G$ act on itself by left multiplication and let $t\in T$ be fixed. Then an action map $x=(m,\sigma(1))$ from $(G,G)$ to $(T,\Gamma)$ can be defined with $m(g)=g^{-1}t$. Then $f(1,x)=\sigma(t)$ which is a group homomorphism. So every $t\in T$ arises from a regular $(1,x)$.

Finally, suppose that $(a_0,x)$ is regular where $x=(n,s)$ is an action map from $(A,H)$ to $(T,\Gamma)$. Then $n:A\to T$, $s\in\Gamma(H)$ and $n(s\gamma a)=\gamma na$ for all $a\in A$, $\gamma\in\Gamma$. Let $t=n(a_0)$ and let $f=f(a_0,x)$. We claim that $\Gamma(f)(s)=\sigma(t)$. Let $H_0$ be the image of $\Gamma$ under the map $s:\Gamma\to H$. Let $\gamma\in\Gamma$ be given and let $h=s(\gamma)$. It suffices to show that if $\gamma t=g^{-1}t$, then $g=f(h)$. But $n(ha_0)=n(s\gamma a_0)=\gamma n(a_0)=\gamma t=g^{-1}t$. Also $n(ha_0)=f(h)^{-1}t$ so $g=f(h)$.

**Proof Theorem 3.3(contd.).** An 'element' of a $G$-Torsor $(T, \sigma)$ in $\mathcal{E}^{\Gamma}$ consists of an element $t \in T$ together with an element $\sigma_t \in \Gamma(G)$. Since elements of $G$-Torsors represent elements of $\hat{\pi}(G)$, we see that $\hat{\pi}(\mathcal{E}^{\Gamma})(G)$ maps nicely to $\hat{\pi}(E)(G) \times \bar{\Gamma}(G)$. It remains to show that this sets up the required isomorphism and the verification is straightforward.

**Corollary 3.6.** $\hat{\pi}(\mathrm{Sets}^{\Gamma}) = \bar{\Gamma}$ *(also* $\bar{\Gamma} = \pi(\mathrm{Sets}^{\Gamma})$*), when $\Gamma$ is a profinite group.*

**Theorem 3.7.** *Let $\mathcal{E}$ be a connected Grothendieck topos and let $p^*: \mathcal{E} \to$ Sets be the inverse image of any point of E. Let $\hat{\pi}(\mathcal{E})$: Fin Grp $\to \mathcal{E}$ as above. Then $p^*\hat{\pi}(\mathcal{E})$ represents a profinite group in* Sets *which is equivalent to the Grothendieck fundamental group of $(\mathcal{E}, p^*)$.*

**Proof.** Let $\mathcal{E}_{\mathrm{lcf}}$ be the category of locally constant finite objects of $\mathcal{E}$ as in [4, pp. 285–290]. Then there is a profinite group $\Gamma$ in Sets for which $\mathcal{E}_{\mathrm{lcf}}$ is isomorphic to $(\mathrm{Fin\ Sets})^{\Gamma}$ in such a way that $p^*$ corresponds to $U^*$ the underlying finite set. For each finite group $G$ the $G$-torsors of $\mathcal{E}$ lie in $\mathcal{E}_{\mathrm{lcf}}$ and so the diagram of $G$-torsors in $\mathcal{E}$ is isomorphic to the diagram of $G$-torsors in $\mathrm{Sets}^{\Gamma}$. Since $\hat{\pi}$ is the colimit of this diagram and since $p^*$ preserves the colimit we see that $p^*\hat{\pi}(\mathcal{E}) = U^*\hat{\pi}(\mathrm{Sets}^{\Gamma}) = \Gamma$ (in the sense that $\Gamma$ is determined by the functor $(\Gamma, -)$ which sends $G$ to the set of continuous homomorphisms from $\Gamma$ to $G$).

**Proposition 3.8.** *Let $\mathcal{E}$ be a topos and let $\hat{\pi} = \hat{\pi}(\mathcal{E})$. Then $\hat{\pi}$ acts in a natural way on every $G$-torsor (for each finite group $G$) so all $G$-torsors live in $\mathcal{E}^{\hat{\pi}}$.*

**Proof.** Let $T$ be a $G$-torsor. As shown in Lemmas 3.4 and 3.5, we need to find an appropriate map $\sigma: T \to \hat{\pi}(G)$ to make $T$ into a $G$-torsor in $\mathcal{E}^{\hat{\pi}}$. But this is immediate as $\hat{\pi}(G)$ is the colimit of all $G$-torsors so there is a coprojection map from $T$ to $\hat{\pi}(G)$ which serves as $\sigma$.

**Remark.** We know that $\hat{\pi}(\mathrm{Sets}^{\Gamma})$ is $\bar{\Gamma}$, a profinite group. As shown in Proposition 3.2, the composition $U^*\bar{\Gamma}$: Fin Grps $\to$ Sets is $\Gamma$ (that is, $U^*\bar{\Gamma}(G) = \mathrm{Con\ Hom}(\Gamma, G)$). There is another important left exact functor, $U_0$, from $\mathrm{Sets}^{\Gamma}$ to Sets where $U_0(A) =$ the $\Gamma$-fixpoint class of $A$. Then $U_0$ is the unique geometric functor from $\mathrm{Sets}^{\Gamma}$ to Sets. (By contrast, $U^*$ is an inverse image functor.) The composition $U_0\bar{\Gamma}$ is then the 'abelianization' of $\Gamma$. [Clearly $U_0\bar{\Gamma}(G) \subseteq U^*\bar{\Gamma}(G)$. Every $f \in U^*\bar{\Gamma}(G)$ is represented by $f: \Gamma \to G$. Then $f$ is a $\Gamma$-fixpoint iff $f(xyx^{-1}) = f(y)$ for all $x, y$ in $\Gamma$ iff $f$ factors through $\Gamma_0$ where $\Gamma_0 = \Gamma/N$, where $N$ is the closure of the commutator subgroup of $\Gamma$.] In all the examples I know of, if $\mathcal{E}$ is a topos over Sets, via $\gamma_*: \mathcal{E} \to$ Sets, then $\gamma_*(\hat{\pi}(\mathcal{E}))$ is abelian. Perhaps there is a reason for it to be the 'first homology group' of $\mathcal{E}$.

**Definition.** Let $\mathcal{E}$ be a connected topos over Sets with geometric functor

$\gamma_*: \mathscr{E} \to$ Sets. Let $\hat{\pi} = \hat{\pi}(\mathscr{E})$. Then $\gamma_*\hat{\pi}$ is a profinite group in Sets (as composing with $\gamma_*$ still yields a left exact functor from Fin Grp). Let $\hat{H}_1(\mathscr{E})$ denote the actual profinite group which represents $\gamma_*\hat{\pi}$ (so that $\gamma_*\hat{\pi}(G) = [\hat{H}_1(\mathscr{E}), G]$ where the bracket denotes the set of continuous homomorphisms). We shall refer to $\hat{H}_1(\mathscr{E})$ as the *first profinite homology group of* $\mathscr{E}$. This is suggested by:

**Theorem 3.9.** *Let* $\mathscr{E}$ *be a connected topos over* Sets. *Then, using the above notation:*

(1) $\hat{H}_1(\mathscr{E})$ *is abelian.*

(2) *For G abelian,* $[\hat{H}_1(\mathscr{E}), G] = H^1(\mathscr{E}, G)$ *as would be predicted by the universal coefficient theorem. (Here* $H^1(\mathscr{E}, G)$ *is defined as in* [4].)

(3) *If* $\mathscr{E}$ *has a point and if* $\hat{\pi}_0$ *is the Grothendieck fundamental profinite group, then* $\hat{H}_1(\mathscr{E})$ *is the abelianization of* $\hat{\pi}_0$ *(the quotient of* $\hat{\pi}_0$ *by the closure of its commutator subgroup).*

**Proof.** We need the following lemmas:

**Lemma 3.10.** *Let T be a G-torsor in a connected topos E where G is a constant, finite group. If T is the coproduct of nontrivial objects A and B, then both A and B have* global extent *(meaning that* $A \to 1$ *and* $B \to 1$ *are epi).*

**Proof.** Let $U \subseteq 1$ and $V \subseteq 1$ be the extent of $A$ and $B$ respectively so that $A \to U$ and $B \to V$ are epi. For each $g \in G$ let:

$$Ag = \{a \in A \mid ga \in A\}.$$

Then the extent of $\bigcap\{Ag \mid g \in G\}$ is a complement for $V$ so either $V = 1$ (and $U = 1$) (in which case we are finished) or $U \cap V = 0$ in which case $U$ or $V$ is 0, contradicting that $A, B$ are non-zero.

**Lemma 3.11.** *Given the above hypotheses, we can write T as a coproduct of subsets,* $T = A_1 + A_2 + \cdots + A_k$ *where each* $A_i$ *is connected and of global extent. Moreover, this decomposition is essentially unique.*

**Proof.** Any object $T$ satisfying the last sentence of the above lemma and also having 'at most $n$ elements' (i.e., satisfies $\forall x_1, \ldots, x_{n+1}, \bigvee(x_i = x_j) \mid i < j\})$ can be so decomposed by induction on $n$.

**Lemma 3.12.** *Given the above hypotheses, there exists a subgroup* $G_0 \subseteq G$ *and a connected* $G_0$*-torsor* $T_0$ *for which T is equivalent to* $G \otimes T$ *(over* $G_0$*).*

**Proof.** Let $T = A_1 + A_2 + \cdots + A_k$. *Let* $T_0 = A_1$ and let $G_0$ be the set of all $g \in G$ for which $g(A_1) = A_1$. (Note either $g(A_1) = A_1$ or $g(A_1)$ misses $A_1$.)

**Proof of Theorem 3.9(contd.).** (1) Recall that $\gamma_*$ is the global section functor, Hom(1, −). Let $G$ be given. We must show that each member of $\hat{H}_1(\mathscr{E})(G)$ arises from $\hat{H}_1(\mathscr{E})(G_0)$ where $G_0$ is an abelian subgroup of $G$. If $s \in \hat{H}_1(\mathscr{E})(G)$ then $s: 1 \to \hat{\pi}(G)$. For each $G$-torsor $T$ let $T^{\#}$ be the colimit of the external diagram consisting of $T$ an all $G$-automorphisms of $T$. If $\{T_i\}$ ranges over a representative set of $G$-torsors, then $\hat{\pi}(G)$ is the coproduct of $T_i^{\#}$. Since $\mathscr{E}$ is connected, $\exists i$ for which $s: 1 \to T_i^{\#}$. [Let $A_i \subseteq 1$ be the truth of $s \in T_i^{\#}$, then the $A_i$'s cover 1 and are pairwise disjoint, so each $A_i$ is complemented, by $\bigcup\{A_j \mid j \neq i\}$, etc.] By Lemma 1.15, $s$ factors through $1 \to \hat{\pi}(G_0)$ where $G_0 \subseteq G$ and $s: 1 \to T_0^{\#}$ where $T_0$ is a connected $G_0$-torsor. (Note that $T_0^{\#} \subseteq \hat{\pi}(G_0)$ and $T_0^{\#}$ maps onto $T^{\#} \subseteq \hat{\pi}(G)$ under $\hat{\pi}(i)$ where $i: G_0 \to G$ and the range of $s$ is contained in $T^{\#}$.) Now let $\theta: T_0 \to T_0$ be $G_0$-equivariant. Then $g \in G_0$ such that $\theta(t) = gt$ for all $g$, as $T_0$ is connected, and this implies that $g$ is in the center $C_0$ of $G_0$. So if $g \notin C_0$, then $gs: 1 \to T_0^{\#}$ is another section, contradicting the connectedness of $T_0$. So $C_0 = G_0$ and $G_0$ is abelian, as required.

(2) If $G$ is abelian, then [4, p. 275], $H^1(\mathscr{E}, G)$ is defined as the set of equivalence classes of $G$-torsors, but, in view of the above proof, this is $\gamma_*\hat{\pi}(\mathscr{E})(G)$, since the $G$-automorphisms act transitively on $T$ so $T^{\#} = 1$, when $G$ is abelian.

(3) In this case the locally constant finite objects of $\mathscr{E}$ can be embedded in $\text{Sets}^{\hat{\pi}_0}$ as in the proof of Theorem 3.7. Now the Remark following Proposition 3.8 applies.

## 4. Examples

### 4.1. Spatial topoi with universal covers

Let $X$ be a locally connected topological space, and let $p: X^* \to X$ be a universal cover. (This means that if $q: Y \to X$ is any other cover and if $x \in X^*$, $y \in Y$ are chosen with $p(x) = g(y)$, then there is a unique $\theta: X^* \to Y$ for which $q\theta = p$ and $\theta(x) = y$.) Let $\pi_0$ be the deck translation group of all isomorphisms $r: X^* \to X^*$ for which $pr = p$. Clearly $\pi_0$ acts transitively on each stalk of $X^*$. We shall interpret $\pi_0$ as a pro-group in Shv($X$) and show that it then corresponds to $\pi$(Shv $X$). We shall construct a sheaf of groups, $\pi$, on $X$ with each stalk isomorphic to $\pi_0$. Define an equivalence relation $E$ on the sheaf $X^* \times X^*$ so that $(x, y)E(a, b)$ iff there exists $r \in \pi_0$ with $(x, y) = (ra, rb)$. Let $\pi$ be the sheaf $(X^* \times X^*)/E$. Then $\pi$ is a group under the operation:

$$(x, y)(a, b) = (rx, b) \quad \text{where } r \in \pi_0 \text{ and } ry = a.$$

It is readily shown that this operation respects $E$-equivalence and it is a group operation with identity $(x, x)$ and with $(x, y)^{-1} = (y, x)$. On each stalk it is clearly isomorphic to $\pi_0$ (by choosing a base point).

To show that $\pi = \pi(\text{Shv } X)$ let $T$ be any $G$-torsor. Then $T$ is a covering of $X$ (for if $\tau: U \to T$ is a local section where $U$ is connected and open, then the sections $\{g\tau \mid g \in G\}$ correspond precisely to the part of $T$ that lies over $U$). Define a map

$$M: T \times X^* \times X^* \to G \quad \text{(the product is taken over } X\text{)}$$

so that $M(t,x,y)=g$ when $\theta: X^* \to T$ is the map over $X$ with $\theta(x)=t$ and $g \in G$ is determined by $\theta(y)=gt$. It is easily checked that $M(t,x,y)=M(t,rx,ry)$ for $r \in \pi_0$, so $M$ is, in effect, a map from $T$ to $G^\pi$, in fact from $T$ to $\mathrm{Hom}(\pi, G)$. In this way $\mathrm{Hom}(\pi, G)$ is readily shown to be the colimit of all $G$-torsors so $\pi = \pi(\mathrm{Shv}\ X)$. [The details are sketched. If $\alpha: \pi \to G$ is given locally, regard $\alpha$ as a map $X^* \times X^* \to G$ over some open $U \subseteq X$. We must construct a $G$-torsor $T$ and find an element $t \in T$ over $U$ for which $\alpha(x,y)=M(t,x,y)$. Choose $x_0 \in U$ and $x \in p^{-1}(x_0) \subseteq X^*$ and regard $\alpha$ as a group homomorphism $\bar{\alpha}: \pi_0 \to G$ with $\bar{\alpha}(r)=\alpha(x,rx)$. Let $T=G \otimes X^*$ (over $\pi_0$) and let $t=1 \otimes x$. Similarly if $t_0 \in T_0$ and $t_1 \in T_1$ produce locally equivalent maps from $\pi$ to $G$ then we can represent $T_0$ and $T_1$ as the same quotient of $X^*$.]

**Remarks.** (1) In the above case, $\pi$ is actually an internal group in $\mathrm{Shv}(X)$. If $\gamma_*: \mathrm{Shv}(X) \to \mathrm{Sets}$ is the global section functor, then the pro-group $\gamma_*[\mathrm{Hom}(\pi, G)]$ is an internal group in Sets, namely, by 3.9, the 'abelianization' of $\pi_0$, or $\pi_0$ modulo its commutator subgroup. To see this, let $f: X^* \times X^* \to G$ be a global group homomorphism which preserves $E$. Suppose $f(x,rx)=g \in G$, then $f(y,ry)=g$ for all $y$ (as the set of all $y$ with this property is clopen). So $f$ gives rise to a function $\bar{f}: \pi_0 \to G$ where $f(x,y)=\bar{f}(s)$ when $y=sx$. Note that $f(x,rx)=\bar{f}(r)$ and $f(sx,srx)=\bar{f}(srs^{-1})$ but $f(x,rx)=f(sx,srx)$ as $f$ respects $E$. So $\bar{f}(r)=\bar{f}(srs^{-1})$. This shows that $\bar{f}$ is a group homomorphism and that

$$\gamma_* \mathrm{Hom}(\pi, G) = \mathrm{Hom}(\pi_0/[\pi_0, \pi_0], G).$$

(2) The pro-group $\pi$ in the above case is represented by an internal group which has also been denoted by $\pi$, so $\pi(G)=\mathrm{Hom}(\pi, G)$. This identification seems natural, but might cause a problem because $\gamma_*\pi$ has two reasonable interpretations and $\gamma_* \mathrm{Hom}(\pi, G)$ is not the same as $\mathrm{Hom}(\gamma_*\pi, G)$. In fact, one is represented by $\pi_0/[\pi_0, \pi_0]$, the other by the center of $\pi_0$ (see *Remark* after 3.8).

### 4.2. The pro-covering

A pro-group in $\mathrm{Shv}(X)$ need not be represented by an internal group, but might be represented by a topological group over $X$ which may fail to a sheaf (cf. Example 6.5 of [7]). This is the case when $X$ is connected and locally path connected, the *universal pro-covering* $X^*$ of $X$ is the inverse limit of all (pointed) connected coverings of $X$ (see [6] and [9]). In general, $X^*$ is a fibration over $X$ but need not have discrete fibres unless it is a cover. The construction of 4.1 can still be applied. (Details from [6] and [9] are needed.)

### 4.3. Totally disconnected spaces

A totally disconnected space may have non-trivial torsors. For example, Heath's Vee Space has a 2 to 1 cover which is a non-trivial $Z_2$-torsor – see [3], [5], and [11]. The proof is by 'categorical topology as practiced by Baire.') Nonetheless, every torsor of a *basically* disconnected space (a space with a clopen base) is trivial over a clopen neighbourhood of any point. So $\hat{\pi}$ and $\pi$ are trivial for such spaces, which includes the Heath Vee Space as it is easily shown to be basically disconnected.

### 4.4. Missed loops, relative groups

The fundamental group of a topos often fails to discern apparent 'loops' because these loops do not lead to torsors. These are connected, locally arc-connected topological spaces with non-trivial fundamental group (in the traditional sense) but which admit no non-trivial coverings, hence no non-trivial torsors (for ordinary groups) (see [6], such spaces are of course not locally simply connected, there is also an example called Schanuel's topos).

Similarly let $C$ be the category with two objects 0 and 1 and maps $m:0\to 1$ and $n:\to 1$ with $nm=m$ and $n^2=1$. Then the topos Fun($C$, Sets) appears to have a 'loop' around 1, but because there are no non-trivial $G$-torsors, for any group $G$, the fundamental groups are trivial. There are not even any torsors for internal groups.

The topos Fun($C^{\mathrm{op}}$, Sets) also has no non-trivial $G$-torsors for constant groups $G$, but does have non-trivial torsors for internal groups. Nonetheless both fundamental groups are trivial. The topoi Fun($C$, Sets) and Fun($C^{\mathrm{op}}$, Sets) are both connected Grothendieck topoi with points. So the Grothendieck fundamental group is defined, and must be trivial.

It might be possible to get at loops such as the ones above by defining the fundamental group of a topos $\mathscr{E}$ relative to a topos $\mathscr{F}$. So if $\mathscr{F}$ is Sets we get our fundamental group; if $\mathscr{F}$ is the topos of finite sets, the profinite group results (and analogously for restrictions to sets of cardinal below $m$).

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